UNIVERSITY OF SASKATCHEWAN

MATHEMATICS 124.3 — Final Examination(April 15, 2000) Solutions

NO BOOKS, NOTES OR CALCULATORS ALLOWED.

Show all of your work. No credit will be given for unsubstantiated correct answers.

(1)(4×2%) Evaluate

(a) $\lim\limits_{x \to +\infty} e^{-\frac{1}{x}}$ **Solution:** $\lim\limits_{x \to +\infty} e^{-\frac{1}{x}} = e^{\lim\limits_{x \to +\infty} -\frac{1}{x}} = e^0 = 1$

(b) $\lim\limits_{x \to 0^-} xe^{-\frac{1}{x}}$ **Solution:** $\lim\limits_{x \to 0^-} xe^{-\frac{1}{x}} = \lim\limits_{x \to 0^-} \frac{e^{-\frac{1}{x}}}{\frac{1}{x}}$ =(by L'Hopital's Rule)

$\lim\limits_{x \to 0^-} \frac{e^{-\frac{1}{x}} \frac{1}{x^2}}{\frac{-1}{x^2}} = \lim\limits_{x \to 0^-} -e^{-\frac{1}{x}} = -\infty$

(c) $\lim\limits_{x \to 0^-} x^x$

Solution: $\lim\limits_{x \to 0^-} x^x = \lim\limits_{x \to 0^-} \left(e^{\ln x}\right)^x = \lim\limits_{x \to 0^-} e^{(x \ln x)} = e^{\left(\lim\limits_{x \to 0^-} x \ln x\right)} = e^{\left(\lim\limits_{x \to 0^-} \frac{\ln x}{\frac{1}{x}}\right)} =$

(by L'Hopital's Rule) $e^{\left(\lim\limits_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)} = e^{\left(\lim\limits_{x \to 0^+} -x\right)} = e^0 = 1$

(d) $\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3}$

Solution: Repeatedly using L'Hopital's Rule, we get:

$$\lim\limits_{x\to 0}\frac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0}\frac{\sec^2 x - \cos x}{3x^2} =$$

$$\lim\limits_{x\to 0}\frac{2\sec^2 x \tan x + \sin x}{6x} = \lim\limits_{x\to 0}\frac{(2\sec^3 x + 1)\sin x}{6x} =$$

$$\lim\limits_{x\to 0}\frac{2\sec^3 x + 1}{6}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{2\sec^3 x + 1}{6}\lim\limits_{x\to 0}\frac{\sin x}{x} =$$

$$\frac{2+1}{6}(1) = \frac{1}{2}$$

(2)(3×3%) Find the derivative $f'(x)$ if $f(x) =$

(a) $\displaystyle\int_{\sin x}^{\cos x} \ln(2 + t)\,dt$ **Solution:** Recalling the formula

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} k(t)\,dt\right) = k(h(x))h'(x) - k(g(x))g'(x),$$

we let $k(t) = \ln(2 + t), g(x) = \sin x$ and $h(x) = \cos x$ and have $f'(x) = \ln(2 + \cos x)(\cos x)' - \ln(2 + x)(\sin x)' = \ln(2 + \cos x)(-\sin x) - \ln(2 + x)(\cos x) =$

$$-\sin x \ln(2 + \cos x) - \cos x \ln(2 + x)$$

(b) $\tanh\left(\ln\frac{1}{x}\right) + \coth\left(\ln\frac{1}{x}\right)$. **Solution:**

Simplify first: Remembering that $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, and that $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ we have

$$\tanh\left(\ln\frac{1}{x}\right) + \coth\left(\ln\frac{1}{x}\right) = \frac{\frac{1}{x} - x}{\frac{1}{x} + x} + \frac{\frac{1}{x} + x}{\frac{1}{x} - x} = \frac{1 - x^2}{1 + x^2} + \frac{1 + x^2}{1 - x^2} = \frac{(1 - x^2)^2 + (1 + x^2)^2}{1 - x^4} =$$

$$\frac{1 - 2x^2 + x^4 + 1 + 2x^2 + x^4}{1 - x^4} = 2\frac{1 + x^4}{1 - x^4},$$

so $f'(x) = 2\dfrac{(1 - x^4)(4x^3) - (1 + x^4)(-4x^3)}{(1 - x^4)^2} = 8x^3\dfrac{1 - x^4 + 1 + x^4}{(1 - x^4)^2} = 16\dfrac{x^3}{(1 - x^4)^2}$

(c) $\sin^{-1}\left(e^{x^2}\right)$. **Solution:** Using $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1 - x^2}}$ and the Chain Rule, we get:

$$\frac{d}{dx}\left(\sin^{-1}\left(e^{x^2}\right)\right) = \frac{1}{\sqrt{1 - \left(e^{x^2}\right)^2}}\left(e^{x^2}\right)' = \frac{1}{\sqrt{1 - e^{2x^2}}}e^{x^2}\left(x^2\right)' = \frac{2xe^{x^2}}{\sqrt{1 - e^{2x^2}}}$$

(3)(4%) Use differentials to estimate the value of $\sin 31°$. **Solution:** If $y = \sin x$,

$dy = \cos x \, dx$, so $\sin 31° = \sin 30° + \cos 30°\dfrac{\pi}{180} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\dfrac{\pi}{180} = \dfrac{1}{2} + \dfrac{\sqrt{3}\pi}{360}$

(4)(2×4%) Evaluate (a) $\sum\limits_{i=1}^{5} \csc\left(\frac{\pi}{6}i\right)$ **Solution:** $\sum\limits_{i=1}^{5} \csc\left(\frac{\pi}{6}i\right) =$

$\csc\left(\frac{\pi}{6}(1)\right) + \csc\left(\frac{\pi}{6}(2)\right) + \csc\left(\frac{\pi}{6}(3)\right) + \csc\left(\frac{\pi}{6}(4)\right) + \csc\left(\frac{\pi}{6}(5)\right) =$

$\frac{1}{\sin\frac{\pi}{6}} + \frac{1}{\sin\frac{\pi}{3}} + \frac{1}{\sin\frac{\pi}{2}} + \frac{1}{\sin\frac{2\pi}{3}} + \frac{1}{\sin\frac{5\pi}{6}} = \frac{1}{\frac{1}{2}} + \frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{1} + \frac{1}{\frac{\sqrt{3}}{2}} + \frac{1}{\frac{1}{2}} = 2 + \frac{2}{\sqrt{3}} + 1 + \frac{2}{\sqrt{3}} + 2 =$

$5 + 2\frac{2}{\sqrt{3}} = \; 5 + 4\frac{\sqrt{3}}{3}$

(b) $\sum\limits_{j=1}^{55} 4j + 1 + \sin\left(\frac{\pi}{2}j\right)$

Solution: $\sum\limits_{j=1}^{55} 4j + 1 + \sin\left(\frac{\pi}{2}j\right) = 4\sum\limits_{j=1}^{55} j + \sum\limits_{j=1}^{55} 1 + (1+0+(-1)+0) + (1+0+(-1)+0) +$

$\cdots + (1+0+(-1)) = 4\frac{55(55+1)}{2} + 55 + 0 = 2(55)(56) + 55 = 55[112+1] = 55(113) =$

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(5)(5%) The interval $[0, \pi]$ is partitioned into the intervals $\left[0, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and $\left[\frac{2\pi}{3}, \pi\right]$. Evaluate the Riemann sum of $f(x) = \cos x$ if the tag or evaluation points are $\frac{\pi}{4}$, $\frac{\pi}{2}$ and $\frac{5\pi}{6}$. Find the exact value of $\int_0^\pi \cos x\, dx$ and compute the error in the approximate value given by the Riemann sum.

Solution: $\mathcal{R}_3(f) = \frac{\pi}{3}\left[\cos\frac{\pi}{4} + \cos\frac{\pi}{2} + \cos\frac{5\pi}{6}\right] = \frac{\pi}{3}\left[\frac{\sqrt{2}}{2} + 0 - \frac{\sqrt{3}}{2}\right] =$

$$\frac{(\sqrt{2} - \sqrt{3})\pi}{6}$$

$\int_0^\pi \cos x\, dx = 0$, so the error is

$$\frac{(\sqrt{2} - \sqrt{3})\pi}{6} \doteq -0.16$$

(6)(5×2%) Evaluate the following definite integrals:

(a) $\int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$ **Solution:**   Let $u = \sin^{-1} x$ so that $du = \frac{1}{\sqrt{1-x^2}}\, dx$. Then

$$\int_{x=0}^{x=\frac{\sqrt{3}}{2}} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx = \int_{u=0}^{u=\frac{\pi}{3}} u\, du = \frac{u^2}{2}\Bigg|_{u=0}^{u=\frac{\pi}{3}} = \frac{\left(\frac{\pi}{3}\right)^2}{2} = \frac{\pi^2}{18}$$

(b) $\int_0^{\frac{\pi}{4}} \tan^5 x \sec^7 x \, dx$

Solution: $\int_{x=0}^{x=\frac{\pi}{4}} \tan^5 x \sec^7 x \, dx = \int_{x=0}^{x=\frac{\pi}{4}} \left(\tan^2 x\right)^2 \sec^6 x \left(\sec x \tan x \, dx\right) =$

$\int_{x=0}^{x=\frac{\pi}{4}} \left(1 - \sec^2 x\right)^2 \sec^6 x \left(\sec x \tan x \, dx\right) = $ (letting $u = \sec x$)

$\int_{u=1}^{u=\frac{2}{\sqrt{2}}} \left(1 - u^2\right)^2 u^6 \, du = \int_{u=1}^{u=\sqrt{2}} \left(1 - 2u^2 + u^4\right) u^6 \, du = \int_{u=1}^{u=\sqrt{2}} \left(u^6 - 2u^8 + u^{10}\right) du =$

$\frac{u^7}{7} - 2\frac{u^9}{9} + \frac{u^{11}}{11} \Big|_{u=1}^{u=\sqrt{2}} = \left(\frac{(\sqrt{2})^7}{7} - 2\frac{(\sqrt{2})^9}{9} + \frac{(\sqrt{2})^{11}}{11}\right) - \left(\frac{1^7}{7} - 2\frac{1^9}{9} + \frac{1^{11}}{11}\right) =$

$\left(\frac{8\sqrt{2}}{7} - 2\frac{16\sqrt{2}}{9} + \frac{32\sqrt{2}}{11}\right) - \left(\frac{1}{7} - 2\frac{1}{9} + \frac{1}{11}\right) = 8\sqrt{2}\left(\frac{1}{7} - \frac{4}{9} + \frac{4}{11}\right) - \left(\frac{1}{7} - \frac{2}{9} + \frac{1}{11}\right) =$

$8\sqrt{2}\left(\frac{99}{693} - \frac{308}{693} + \frac{252}{693}\right) - \left(\frac{99}{693} - \frac{154}{693} + \frac{63}{693}\right) = 8\sqrt{2}\left(\frac{47}{693}\right) - \left(\frac{8}{693}\right) = 8\dfrac{47\sqrt{2} - 1}{693}$

(c) $\int_0^{\frac{\pi}{3}} \sin^2 2x \, dx$ **Solution:** $\int_0^{\frac{\pi}{3}} \sin^2 2x \, dx = \int_0^{\frac{\pi}{3}} \frac{1 - \cos 4x}{2} \, dx = \frac{1}{2}\int_0^{\frac{\pi}{3}} 1 - \cos 4x \, dx =$

$\frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right)\Big|_0^{\frac{\pi}{3}} = \frac{1}{2}\left(\frac{\pi}{3} - \frac{1}{4}\sin 4\frac{\pi}{3}\right) = \frac{1}{2}\left[\frac{\pi}{3} - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{16}$

(d) $\int_0^\pi x^2 \sin x \, dx$ **Solution:** Use Integration by Parts:

let $u = x^2$ and $dv = \sin x \, dx$ so that $du = 2x \, dx$ and $v = -\cos x$. Then we have:

$$\int_{x=0}^{x=\pi} x^2 \sin x \, dx = \int_{x=0}^{x=\pi} u \, dv = uv\big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} v \, du = x^2(-\cos x)\big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} (-\cos x)(2x \, dx) =$$

$$-x^2 \cos x \big|_{x=0}^{x=\pi} + 2 \int_{x=0}^{x=\pi} x \cos x \, dx = \text{(letting } U = x \text{ and } dV = \cos x \, dx)$$

$$-\pi^2 \cos \pi + 2 \left(\int_{x=0}^{x=\pi} U \, dV \right) = -\pi^2(-1) + 2 \left(UV\big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} V \, dU \right) =$$

$$\pi^2 + 2 \left(x \sin x \big|_{x=0}^{x=\pi} - \int_{x=0}^{x=\pi} \sin x \, dx \right) = \pi^2 + 2 \left(-\cos x \big|_{x=0}^{x=\pi} \right) = \pi^2 + 2 \left(-(-1) - (-1) \right) =$$

$$\pi^2 + 4$$

(e) $\int_1^2 \frac{1}{x^3 + 4x} \, dx$ **Solution:** Use Partial Fractions: $\frac{1}{x^3 + 4x} = \frac{1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$, so we must have

$$1 = A(x^2 + 4) + x(Bx + C).$$

Letting $x = 0$, we get $1 = A(0^2 + 4) + 0(B(0) + C)$ or $1 = 4A$, so $A = \frac{1}{4}$.

Letting $x = 2i$, we get

$$1 = A((2i)^2 + 4) + 2i(B(2i) + C), \text{ or}$$

$$1 = A(-4 + 4) + -4B2 + 2Ci \text{ or}$$

$$1 = -4B + 2Ci \text{ so}$$

$B = -\frac{1}{4}$ and $C = 0$. Thus:

$$\frac{1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} = \frac{\frac{1}{4}}{x} + \frac{-\frac{1}{4}x + 0}{x^2 + 4} = \frac{1}{4}\frac{1}{x} - \frac{1}{4}\frac{x}{x^2 + 4}, \text{ so}$$

$$\int_1^2 \frac{1}{x^3 + 4x} \, dx = \frac{1}{4} \int_1^2 \frac{1}{x} \, dx - \frac{1}{8} \int_1^2 \frac{2x}{x^2 + 2^2} \, dx =$$

$$\frac{1}{4}\ln x\Big|_1^2 - \frac{1}{8}\ln\left(x^2 + 2^2\right)\Big|_1^2 = \frac{1}{4}\ln 2 - \frac{1}{8}\left(\ln\left(2^2 + 2^2\right) - \ln\left(1^2 + 2^2\right)\right) =$$

$$\frac{1}{4}\ln 2 - \frac{1}{8}(\ln 8 - \ln 5) = \frac{1}{4}\ln 2 - \frac{1}{8}(3\ln 2 - \ln 5) = -\frac{1}{8}\ln 2 + \frac{1}{8}\ln 5 = \frac{1}{8}\ln\frac{5}{2}$$

(7)(4×2%) Evaluate the following indefinite integrals:

(a) $\int \frac{x+1}{\sqrt[3]{x^4}}dx$ **Solution:** $\int \frac{x+1}{\sqrt[3]{x^4}}dx = \int \frac{x}{x^{\frac{4}{3}}} + \frac{1}{x^{\frac{4}{3}}}dx = \int x^{-\frac{1}{3}} + x^{-\frac{4}{3}}dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} +$

$\frac{x^{-\frac{1}{3}}}{-\frac{1}{3}} + C =$

$$\frac{3}{2}x^{\frac{2}{3}} - 3x^{-\frac{1}{3}} + C$$

(b) $\int \frac{2x+8}{x^2+4x+6}dx$ **Solution:** $\int \frac{2x+8}{x^2+4x+6}dx = \int \frac{2x+4+4}{x^2+4x+6}dx =$

$\int \frac{2x+4}{x^2+4x+6}dx - \int \frac{4}{(x+2)^2 + (\sqrt{2})^2}dx =$

$$\ln(x^2 + 4x + 6) + \frac{4}{\sqrt{2}}\arctan\frac{x+2}{\sqrt{2}} + C$$

(c) $\int \dfrac{dx}{x^2\sqrt{4+x^2}}$ **Solution:** Use the Trigonometric Substitution $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, so that

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{(2\tan\theta)^2\sqrt{4+(2\tan\theta)^2}} = \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{2^2(1+\tan^2\theta)}}d\theta =$$

$$\frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}d\theta = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta =$$

$$\frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = (\text{ letting } u = \sin\theta)$$

$$\frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\frac{u^{-1}}{-1} + C = -\frac{1}{4}(\sin\theta)^{-1} + C = -\frac{1}{4\sin\theta} + C = -\frac{1}{4\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}} + C =$$

$$-\frac{\sqrt{1+\tan^2\theta}}{4\tan\theta} + C = -\frac{\sqrt{1+\left(\frac{x}{2}\right)^2}}{4\frac{x}{2}} + C = -\frac{\sqrt{1+\frac{x^2}{4}}}{2x} + C = -\frac{\sqrt{4+x^2}}{4x} + C$$

where we have used the identity $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

(d) $\int \cos 8x \cos 4x \, dx$ **Solution:** Adding the basic identities

$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ and
$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, we get

$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$, so

$\cos \alpha \cos \beta = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$, and thus

$\cos 8x \cos 4x = \frac{1}{2} \cos(8x + 4x) + \frac{1}{2} \cos(8x - 4x) = \frac{1}{2} \cos 12x + \frac{1}{2} \cos 4x.$

Thus we have

$\int \cos 8x \cos 4x \, dx = \frac{1}{2} \int \cos 12x \, dx + \frac{1}{2} \int \cos 4x \, dx = \frac{1}{2} \left(\frac{1}{12} \sin 12x \right) + \frac{1}{2} \left(\frac{1}{4} \sin 4x \right) + C =$

$$\frac{1}{24} \sin 12x + \frac{1}{8} \sin 4x + C.$$

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(8)(2×2%) Evaluate the following improper integrals:

(a) $\int_0^2 \frac{dx}{x^2 - x - 2}$ **Solution:** Using the Method of Partial Fractions,

$\frac{1}{x^2 - x - 2} = \frac{1}{(x - 2)(x + 1)} = \frac{A}{x + 1} + \frac{B}{x - 2}$, or

$1 = A(x - 2) + B(x + 1).$

Substituting $x = 2$ gives $1 = A(2 - 2) + B(2 + 1)$, so $B = \frac{1}{3}$,

and substituting $x = -1$ gives $1 = A(-1 - 2) + B(-1 + 1)$, so $A = -\frac{1}{3}$.

Therefore

$\int \frac{dx}{x^2 - x - 2} = \int \frac{-\frac{1}{3}}{x + 1} dx + \int \frac{\frac{1}{3}}{x - 2} dx = -\frac{1}{3} \ln|x + 1| + \frac{1}{3} \ln|x - 2| + C = \frac{1}{3} \ln \left| \frac{x - 2}{x + 1} \right| + C$, so

$\int_0^2 \frac{dx}{x^2 - x - 2} = \lim_{T \to 2^-} \int_0^T \frac{dx}{x^2 - x - 2} = \lim_{T \to 2^-} \frac{1}{3} \ln \left| \frac{x - 2}{x + 1} \right| \Big|_0^T = $ which **DIVERGES** to $-\infty$
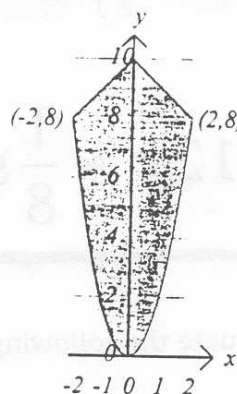
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(b) $\int_1^\infty \dfrac{dx}{(3x+1)^3}$ Solution:

$$\int_1^\infty \frac{dx}{(3x+1)^3} = \lim_{T\to\infty} \int_1^T \frac{dx}{(3x+1)^3} = \lim_{T\to\infty} \frac{1}{3} \frac{(3x+1)^{-2}}{-2} \Big|_1^T =$$

$$\lim_{T\to\infty} -\frac{1}{6(3x+1)^2}\Big|_1^T = \lim_{T\to\infty} -\frac{1}{6(3T+1)^2} - \left(-\frac{1}{6(3(1)+1)^2}\right) = 0 + \frac{1}{6(4^2)} = \frac{1}{96}$$

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(9) Let $\mathcal{R}$ be the region consisting of all points which lie above the curve $y = 2x^2$ AND below the curve $y = 10 - |x|$.

(a)(2%) Sketch $\mathcal{R}$.   Solution:

Do NOT evaluate any of the integrals in the following questions.
(b)(1%) Express the area of $\mathcal{R}$ as a definite integral.

Solution: $A = \displaystyle\int_{-2}^{2} \left[(10 - |x|) - 2x^2 \right] dx$

(c)(1%) Express the length of the perimeter of $\mathcal{R}$ in terms of definite integrals.
Solution:

$$L = 2\int_0^2 \sqrt{1 + (-1)^2}\, dx + 2\int_0^2 \sqrt{1 + (4x)^2}\, dx$$

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(d)(1%) Express the area of the surface obtained by rotating $\mathcal{R}$ about the x-axis as a definite integral.

Solution:

$$S = \int_{-2}^{2} 2\pi \left[(10 - |x|)\sqrt{5} + 2x^2\sqrt{1 + (4x)^2} \right] dx$$

(e)(1%) Express the volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis as a definite integral.

Solution: $\displaystyle V = \int_{-2}^{2} (10 - |x|)^2 - (2x^2)^2 \, dx$

(f)(2%) Express the volume of the solid obtained by rotating $\mathcal{R}$ about the y-axis as a definite integral.

Solution: $\displaystyle V = \int_{0}^{2} 2\pi x \left[10 - x - 2x^2 \right] dx$

(g)(3%) Express the centroid of $\mathcal{R}$ in terms of definite integrals.

Solution: $\displaystyle C = \left(\frac{\int_{-2}^{2} x \left[(10 - |x|) - 2x^2 \right] dx}{\int_{-2}^{2} \left[(10 - |x|) - 2x^2 \right] dx}, \frac{\frac{1}{2}\int_{-2}^{2} \left[(10 - |x|)^2 - (2x^2)^2 \right] dx}{\int_{-2}^{2} \left[(10 - |x|) - 2x^2 \right] dx} \right)$

(10)(2%) Express the area of the surface obtained by rotating the curve $x = t^3$, $y = t^4$, $0 \le t \le 1$ about the x-axis as a definite integral. Do NOT evaluate the integral.

Solution: $\displaystyle S = \int_{0}^{1} 2\pi(t^4)\sqrt{(3t^2)^2 + (4t^3)^2} \, dt = 2\pi \int_{0}^{1} t^4\sqrt{9t^4 + 16t^6} \, dt$
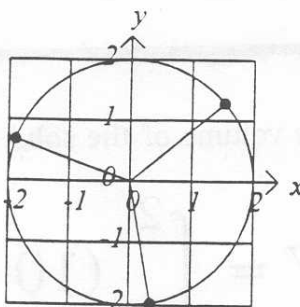
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(11) (a)(3%) Find all cube (or third) roots of the complex number $-4 + 4\sqrt{3}i$.

Solution: $-4 + 4\sqrt{3}i = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$, so the the three roots are:

$$2\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right), \; 2\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right), \; 2\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right),$$
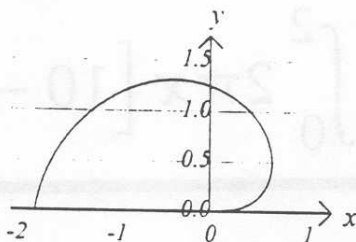
(b)(1%) Sketch their locations on a graph.

Solution:

(12) (a)(3%) Sketch the curve whose polar equation is $r = \sqrt{\theta}, 0 \le \theta \le \pi$.

Solution:

(b)(2%) Express its length as a definite integral. Do NOT evaluate the integral.

Solution: $L = \int_\alpha^\beta \sqrt{r^2 + \dot{r}^2}\, d\theta = \int_0^\pi \sqrt{\left(\sqrt{\theta}\right)^2 + \left(\frac{1}{2\sqrt{\theta}}\right)^2}\, d\theta = \int_0^\pi \sqrt{\theta + \frac{1}{4\theta}}\, d\theta$

(13)(3%) Consider a function f defined on the interval [0,8]. Some values of $f(x)$ are given in the following table:

x	$f(x)$
0	1.2
1	1.1
2	0.9
3	0.8
4	0.7
5	0.8
6	1.0
7	1.2
8	1.5

Some possibly useful facts derived from the table:

The sum at the left-hand endpoints is
$$f(0) + f(2) + f(4) + f(6) = 3.8,$$

the sum at the right-hand endpoints is
$$f(2) + f(4) + f(6) + f(8) = 4.1, \text{ and}$$

the sum at the midpoints is $f(1) + f(3) + f(5) + f(7) = 3.9.$

Find the Trapezoidal, Midpoint and Simpson's Rule estimates using 4 intervals each for $\int_0^8 f(x)\,dx$ from the information given. **Solution:** We have $\Delta x = 2$, so $\mathcal{L}_4(f) = 2(f(0) + f(2) + f(4) + f(6)) = 2(3.8) = 7.6$, $\mathcal{R}_4(f) = 2(f(2) + f(4) + f(6) + f(8)) =$

$2(4.1) = 8.2$, so $T_4(f) = \dfrac{\mathcal{L}_4(f) + \mathcal{R}_4(f)}{2} = \dfrac{7.6 + 8.2}{2} = \dfrac{15.8}{2} = 7.9$.

$\mathcal{M}_4(f) = 2(f(1) + f(3) + f(5) + f(7)) = 2(3.9) = 7.8.$

finally, $S_8(f) = \dfrac{T_4(f) + 2\mathcal{M}_4(f)}{3} = \dfrac{7.9 + 2(7.8)}{3} = \dfrac{7.9 + 15.6}{3} = \dfrac{23.5}{3} = 7\dfrac{5}{6}$

(14)(3%) A function satisfies the differential equation $y' = ky$. Its graph is also known to pass through the point (2,10) and (10,5). Find expressions for $y(0)$ and $y(12)$.

Solution: We know that $y = y(0)e^{kx}$, where $y(0)$ and k are as yet unknown. Using the given data points, we have

$10 = y(0)e^{k(2)}$ or $10 = y(0)(e^k)^2$ and $5 = y(0)e^{k(10)}$ or $5 = y(0)(e^k)^{10}$, so, dividing the first equation by the second, we get

$\frac{10}{5} = \frac{y(0)(e^k)^2}{y(0)(e^k)^{10}}$ or $2 = (e^k)^{-8}$ which gives us $e^k = 2^{-\frac{1}{8}}$.

Substituting this into the equation $10 = y(0)(e^k)^2$, we get $10 = y(0)\left(2^{-\frac{1}{8}}\right)^2 = y(0)2^{-\frac{1}{4}}$,

so $$y(0) = 10\left(2^{\frac{1}{4}}\right) = 10\sqrt[4]{2}.$$

This gives a formula for $y(t)$:

$$y(t) = 10\sqrt[4]{2}2^{-\frac{1}{8}t} = 10\sqrt[4]{2}2^{-\frac{t}{8}}$$

Thus $$y(12) = 10\sqrt[4]{2}2^{-\frac{12}{8}} = 10\left(2^{\frac{1}{4}}\right)2^{-\frac{3}{4}} = 10\left(2^{-\frac{1}{2}}\right) = 5\sqrt{2}$$

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(15)(3%) Find the solution of the differential equation $e^y y' = \frac{3x^2}{1+y}$ which satisfies the initial condition $y(2) = 0$

Solution: Separating Variables, we get:

$(1+y)e^y dy = 3x^2 dx$ which integrates to $ye^y = x^3 + C$. Using $y(2) = 0$, we get $0e^0 = 2^3 + C$, so $C = -8$. The solution is thus $ye^y = x^3 - 8$. Solving for x, we get

$$x = \sqrt[3]{ye^y + 8}$$

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(16)(3%) A force of 10N is required to hold a spring that has been stretched from its natural length of 10cm to 90cm. How much work is done in stretching the spring from 35 cm to 45 cm?

Solution: We have $F = 10N = k(90 - 10) \text{ cm} = k(80)\text{cm}$, so $k = \frac{10}{80}\frac{N}{\text{cm}} = \frac{10}{0.80}\frac{N}{m} = 12.5\frac{N}{m}$

The work required to extend the spring from 35 to 45 cm is

$$W = \int_{(35-10)\text{cm}}^{(45-10)\text{cm}} 12.5x\frac{N}{m}dx = 12.5\frac{N}{m}\int_{25\text{cm}}^{35\text{cm}} xdx = 12.5\frac{N}{m}\left(\frac{x^2}{2}\right)\bigg|_{25\text{cm}}^{35\text{cm}} =$$

$$6.25\frac{N}{m}\left((0.35\text{m})^2 - (0.25\text{m})^2\right) = 6.25\frac{N}{m}0.06\text{m}^2 = 0.375 \text{ N·m} = \frac{3}{8}\text{N·m}$$

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(17)(5%) A water tank has parallel vertical cross-sections which are parabolas that are 12 metres high and 4 metres wide at the top. The vertices are at the bottom of the tank. If the origin of an xy-coordinate system is placed at the vertex of a cross-section, the equation of the parabola is $y = 3x^2$. If the tank is full, what force does the water exert against an end of the tank?

Solution: First find the y-coordinate of the centroid of the end of the tank. The area is $\mathcal{A} = 2\int_0^2 12 - 3x^2 dx = 2(12x - x^3)\big|_0^2 = 2[12(2) - 2^3] = 32$(square metres).

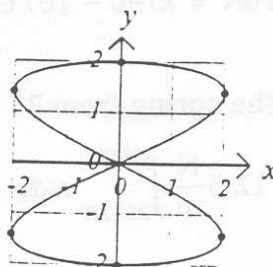The moment about the x-axis is $\mathcal{M}_y = \int_{-2}^2 \frac{1}{2}\left[(12)^2 - (3x^2)^2\right] dx = 2\frac{1}{2}\int_0^2 \left[144 - 9x^4\right] dx =$

$144x - 9\frac{x^5}{5}\bigg|_0^2 = 144(2) - 9\frac{2^5}{5} = 288\frac{4}{5}$, so $\bar{y} = \frac{\mathcal{M}_y}{\mathcal{A}} = \frac{288\frac{4}{5}}{32} = 9\frac{4}{5}$

The force is thus $\mathcal{F} = 9.8(1000)(12-9\frac{4}{5})32 = 9800\frac{24}{5}32$(Newtons).

(18) (a)(3%) Sketch the graph of the curve $x = 2 \sin 2t$, $y = 2 \sin t$, $0 \leq t \leq 2\pi$. Label all points where the tangent is vertical or horizontal.

Solution:

$\dot{x} = 4 \cos 2t$ and $\dot{y} = 2 \cos t$. Vertical tangents occur when $\dot{x} = 0$ and $\dot{y} \neq 0$, i.e., when $t = \dfrac{\pi}{4}$, $t = \dfrac{3\pi}{4}$, $t = \dfrac{5\pi}{4}$, and $t = \dfrac{4\pi}{4}$, or at the points $(2, \sqrt{2})$, $(-2, \sqrt{2})$, $(-2, -\sqrt{2})$, and $(2, -\sqrt{2})$.

Horizontal tangents occur when $\dot{y} = 0$ and $\dot{x} \neq 0$, i.e., when $t = \dfrac{\pi}{2}$, and $t = \dfrac{3\pi}{2}$ or at the points $(0, 2)$, and $(0, -2)$.

(b)(2%) Find a Cartesian equation (an equation in x and y) which has the same graph.

Solution: $\dfrac{y}{2} = \sin t =, \dfrac{x}{2} = 2 \sin t \cos t = y \cos t = y \pm \sqrt{1 - \sin^2 t} = \pm y \sqrt{1 - \left(\dfrac{y}{2}\right)^2} =,$

or

$\dfrac{x}{2} = \pm y \sqrt{1 - \dfrac{y^2}{4}} = \pm y \dfrac{\sqrt{4 - y^2}}{2}$, so

$x = \pm y \sqrt{4 - y^2}$. Squaring, we get:

$$x^2 = y^2 (4 - y^2)$$